A proof of the inconsistency of ZFC

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Abstract

This paper shows that ZFC is not consistent.

1 Introduction

Zermelo-Fraenkel set theory with the axiom of choice (ZFC) serves as a foundation for mathematics Fraenkel et al. [1973].

The purpose of this paper is to show that the theory is not consistent.

Section 2 presents the proof and Section 3 concludes.

2 Proof

To show the inconsistency of ZFC this proof uses primarily one of the axioms of ZFC, the axiom of pairing, to derive a contradiction.

The axiom of pairing states that if any sets x and y exist, then another set exists that contains them. If x and y are the same set, then a set exists that contains that set.

Define a function $f(x)$ such that $f(x)$ is a set containing only x.

$$f(x) = \{x\} \tag{1}$$

ZFC implies that if any set x exists then $f(x)$ exists. Also, ZFC implies that the empty set exists.

$\emptyset$
$\forall x f(x)$

2.1 Defining a domain of discourse

Is it possible to define a domain of discourse which satisfies the above conditions?

Some may argue that a domain of discourse may be defined recursively as follows:

$\emptyset$
$\forall x f(x)$

However, careful consideration of recursion shows that the above conditions do not define a domain of discourse.

The term x above can only refer to one individual at one time. Therefore, the above recursive rule can only be used to define one individual in the domain of discourse at a time.

Over time, many individuals may be defined. There is no bound to the number of individuals that may be defined over time, and in that sense an infinite number of individuals may be defined, one at a time.

The conditions above do not specify when the domain of discourse has been defined. Therefore, the conditions do not define a domain of discourse.

2.2 No domain satisfies

No domain of discourse both includes the empty set and satisfies the condition $\forall x f(x)$.

Suppose that the domain of discourse includes only the empty set.

If so, then the domain of discourse does not satisfy the condition that if x exists then $f(x)$ exists, since the domain of discourse does not include the set containing only the empty set.

The domain of discourse includes all but one set required by the condition.

It is not possible to remedy this situation by including the required set in the domain of discourse.

Including the required set results in a changed domain of discourse, but the domain of discourse still includes all but one set required by the condition.

Therefore by induction, given any domain of discourse containing the empty set, there is at least one x in the domain such that $f(x)$ is not in the domain.

$$\emptyset$$
$$\exists x \neg f(x)$$

2.3 Other methods

There are other ways to show that if the empty set exists then there exists an x such that $f(x)$ does not exist. Consider the following arguments.

For any two individuals x and y in the domain of discourse, if $f(x) = f(y)$ then $x = y$.

$$\forall x \forall y (f(x) = f(y) \implies x = y)$$

In other words, no two distinct individuals in the domain of discourse map to the same individual by means of the function f. The function f is an injection.

Also, there is at least one individual in the domain of discourse, the empty set, which cannot be mapped to by means of the function f.

$$\exists x \neg \exists y (f(y) = x)$$

Since f does not map any two individuals in the domain to the same individual, and since there is at least one individual in the domain which f does not map to, then there must be at least one individual in the domain from which f does not map to any individual in the domain, see Figure 1.

Therefore, the existence of the empty set implies that there exists at least one x such that $f(x)$ does not exist.

Further, assume that it is true that $\forall x f(x)$. Then for every individual in the domain of discourse f maps that individual to a corresponding unique individual in the domain. In other words, f is a bijective function from the domain of discourse to the domain of discourse.

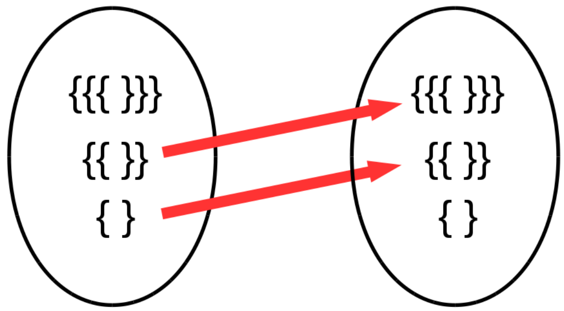

Figure 1: Illustration of the function $f(x)$ mapping from a domain of discourse to the same domain

However, f is not such a bijection because there is one individual in the domain of discourse, the empty set, which cannot be mapped to by means of f.

Therefore by contradiction, it is not true that $\forall x f(x)$, given that the empty set exists. There must be at least one x such that $f(x)$ does not exist.

2.4 Inconsistency

The arguments presented in this proof show that for any given domain of discourse including the empty set there exists an x such that $f(x)$ does not exist.

Yet ZFC implies that the empty set exists and that the existence of any x implies the existence of $f(x)$.

Therefore, ZFC is inconsistent by derivation of the contradiction that the existence of any x implies the existence of $f(x)$ and there exists an x such that $f(x)$ does not exist.

$\emptyset$
$\forall x f(x)$
$\exists x \neg f(x)$
$\neg \forall x f(x)$

3 Conclusion

The proof presented in this paper shows that ZFC is not consistent.

References

Abraham Adolf Fraenkel, Yehoshua Bar-Hillel, and Azriel Levy. *Foundations of set theory*. Elsevier, 1973.